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On wide sense stationary processes over finite non-abelian groups

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Abstract: Let X be a real-valued wide sense stationary process over a finite non-abelian group G. We provide results on optimal orthogonal decomposition of X into real-valued mutually orthogonal components and using this decomposition we develop a test for correlation of X over the group G. Applications of these results to the analysis of variance of the carry-over effects in the cross-over designs in clinical studies are given. Our focus will be on groups S_3, S_4, and A_4.

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1. Introduction

Applications of group representations to probability and statistics is a rich subject with Diaconis (1998) as an excellent reference. In this paper, we will study some aspects of the spectral theory where the underlying group G is finite non-abelian. Please see Giannakis (1999) for abelian case. In particular, we will consider wide sense stationary processes over the group G. We refer the reader to Peccati and Pycke (2005) for material on stochastic processes over non-abelian groups. We will consider finite non-abelian groups, provide simplified proofs of the relevant results in the finite setting, and give results on the optimality of the orthogonal decompositions into real components. We also provide a classification on the case of X being de-correlated over G. Applications of these results to carry-over effects in the cross-over designs in clinical studies will be given later. Our focus will be on groups S_3, S_4, and A_4.

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PUBLIC INTEREST STATEMENT

Applications of group representations to probability and statistics is a rich subject. Our paper connects the representation theory with the study of stochastic processes over non-commutative groups. The main results are applied to carry-over effects in the cross-over designs in clinical studies.

Let G denote a finite non-abelian group and let X denote a zero mean real-valued wide sense stationary process (WSS) over the group G. In particular, we have $E(X(t)) = 0$ for all $t \in G$. The auto-correlation function R_X of X is defined as follows, set $\tau = t^{-1}s$, where $t, s \in G$

$$R_X(t, s) = E(X(t)X(s)) = R_X(\tau).$$

Suppose X and Y are two zero mean WSS processes over G. We define the cross-correlation function of X and Y as follows:

$$R_{XY}(\tau) = E(X(t)Y(t\tau)).$$

Let $\{X_i\}_{i=1}^r$ denote a family of zero mean WSS processes over G. Such a family is said to be mutually orthogonal if $R_{X_iX_j}(\tau) = 0$ for all $\tau \in G$ and $i \neq j$. In such case, we refer to the WSS process over G as white. For more information of random processes in general, we refer the reader to Bartoszynski and Niewiadomska-Bugaj (2008).

A natural place where WSS processes over non-abelian groups arise are the cross-over designs in clinical trials. We will refer to this later on in this paper.

Non-abelian Fourier Transform. Let $\mathbb{C}^n$ denote the n-dimensional vector space over the complex numbers. The standard basis for $\mathbb{C}^n$ is identified with the ordered group elements of G, where $|G| = n$.

A finite dimensional representation of a finite group G over $\mathbb{C}$ is a group homomorphism

$$\rho: G \mapsto GL(d_j, \mathbb{C}),$$

where $GL(d_j, \mathbb{C})$ denotes the general linear group, the set of all $d_j \times d_j$ invertible matrices. We refer to d_j as the degree of the group representation.

Two group representations

$$\rho_1: G \mapsto GL(d_j, \mathbb{C}) \quad \text{and } \rho_2: G \mapsto GL(d_j, \mathbb{C})$$

are said to be equivalent if there exists an $d_j \times d_j$ invertible matrix T such that

$$T \circ \rho_1(g) \circ T^{-1} = \rho_2(g)$$

for all $g \in G$.

An irreducible group representation of G is a group representation ρ of G, for which there is no non-trivial subspace W of $\mathbb{C}^j$ for which

$$\rho(g)W \subset W$$

for all $g \in G$.

Let $\mathbb{C}[G]$ be the algebra of complex-valued functions on G with respect to G-convolution. Let $\psi = (c_0, c_1, \ldots, c_{n-1}) \in \mathbb{C}^n$ and identify the function ψ with its symbol

$$\Psi = c_0 1 + c_1 g_1 + \cdots c_{n-1} g_{n-1} \in \mathbb{C}[G], \quad \text{where } G = \{g_0 = 1, g_1, \ldots, g_{n-1}\}.$$

A G-convolution of ψ and ϕ is defined by the following action, $\sigma \in G$

$$(\psi * \phi)(\sigma) = \sum_{\tau \in G} \psi(\sigma\tau^{-1})\phi(\tau).$$

Let $\widehat{G}$ be the set of all (equivalence classes) irreducible representations of the group G. WLOG we can assume these representations are unitary (please see Hazewinkel (2001) for further details). Let $\rho \in \widehat{G}$ be of degree d_j and let $\phi \in \mathbb{C}^n$. Then, the Fourier transform of ϕ at ρ is the $d_j \times d_j$ matrix

$$\widehat{\phi}(\rho) = \sum_{s \in G} \phi(s)\rho(s).$$

The Fourier inversion formula, $s \in G$, is given by

$$\phi(s) = \frac{1}{|G|} \sum_{\rho_j \in \widehat{G}} d_j \operatorname{tr}\left(\rho_j(s^{-1})\widehat{\phi}(\rho_j)\right).$$

Observe the the switch $s \to s^{-1}$ in the above functions. We refer the reader to Stankovic, Radomir, Moraga, and Astola (2005) for further reading on this subject.

Let ψ and ϕ be two elements in $\mathbb{C}^n$. We have a natural identification

$$\psi * \phi \mapsto \Psi\Phi.$$

The action of ψ on ϕ via G-convolution is delivered by the matrix multiplication by the G-circulant matrix $C_G(\psi)$

$$\psi * \phi = C_G(\psi)\phi.$$

Definition 1 For given vectors $\psi, \phi \in l^2(G)$ the G cross-correlation function is defined by

$$R_{\psi,\phi}(\tau) = \sum_{t \in G} \overline{\psi}(t)\phi(t\tau).$$

We have, see Zizler (2014),

$$R_{\psi,\phi} = \Psi^*\Phi = C_G^*(\psi)\phi.$$

The character of a group representation ρ is the complex-valued function

$$\chi : G \to \mathbb{C}$$

defined by

$$\chi(g) = \operatorname{tr}(\rho(g)),\ g \in G.$$

We call a character irreducible if the underlying group representation is irreducible. We define an inner product on the space of class functions, functions on G that are constant on its conjugacy classes

$$\langle \chi, \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\theta(g)}.$$

A character is a class function. There are as many irreducible characters as there are conjugacy classes of G(please see Dummit (1999), p. 870 for details). Let r denote the number of conjugacy classes of G and we have r irreducible characters $\{\chi_1, \chi_2, \ldots, \chi_r\}$ for the group G. We have, with respect to the usual inner product,

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}, \quad \text{for all } i, j \in \{1, 2, \ldots, r\}$$

where $\delta_{i,j}$ is the Kronecker delta. The set of all irreducible characters form a basis for the space of class functions on G, see Dummit (1999) for more details.

The Fourier transform gives us a natural isomorphism

$$\mathbb{C}[G] \Rightarrow M(\widehat{G}),$$

where

$$M(\widehat{G}) = M_{d_1\times d_1}(\mathbb{C}) \oplus M_{d_2\times d_2}(\mathbb{C}) \oplus \cdots \oplus M_{d_r\times d_r}(\mathbb{C}),$$

with $d_1^2 + d_2^2 + \cdots + d_r^2 = n$. A typical element of $\mathbb{C}^n$ is a complex-valued function

$$\psi = (c_0, c_1, \ldots, c_{n-1})$$

and the typical element of $M(\widehat{G})$ is the direct sum of Fourier transforms

$$\widehat{\phi}(\rho_1) \oplus \widehat{\phi}(\rho_2) \oplus \cdots \oplus \widehat{\phi}(\rho_r).$$

Fourier transform turns convolution into (matrix) multiplication

$$\widehat{\psi * \phi} = \bigoplus_{j=1}^{r} \widehat{\psi}_j \widehat{\phi}_j = \widehat{\psi}\widehat{\phi}.$$

Equip the space $M(\widehat{G})$ with the following inner product. Let $\mathbf{v} = \widehat{\phi}(\rho_1) \oplus \widehat{\phi}(\rho_2) \oplus \cdots \oplus \widehat{\phi}(\rho_r)$ and $\mathbf{w} = \widehat{\zeta}(\rho_1) \oplus \widehat{\zeta}(\rho_2) \oplus \cdots \oplus \widehat{\zeta}(\rho_r)$.

Then

$$\langle \mathbf{v} \bullet_F \mathbf{w} \rangle = \frac{d_1}{|G|} \operatorname{tr}\left(\widehat{\phi}(\rho_1)\widehat{\zeta}^*(\rho_1)\right) + \frac{d_2}{|G|} \operatorname{tr}\left(\widehat{\phi}(\rho_2)\widehat{\zeta}^*(\rho_2)\right) + \cdots + \frac{d_r}{|G|} \operatorname{tr}\left(\widehat{\phi}(\rho_r)\widehat{\zeta}^*(\rho_r)\right)$$

whete $\widehat{\zeta}^*(\rho)$ denotes the adjoint of $\widehat{\zeta}(\rho)$.

Let $\phi \in \mathbb{C}^n$ and define for $s \in G$

$$\phi_j(s) = \frac{d_j}{|G|} \operatorname{tr}\left(\rho_j(s^{-1})\widehat{\phi}(\rho_j)\right).$$

Note $\phi = \sum_{j=1}^{r} \phi_j$. We are able to decompose the function ϕ into a sum of r functions which is the number of conjugacy classes of G.

We define an (orthogonal) projection P_j on $\mathbb{C}^n$ by the following action, $\phi \in \mathbb{C}^n$

$P_j(\phi) = \phi_j$.

The action of the linear operator P_j in the Fourier domain is given by the (matrix) multiplication by the vector

$$0 \oplus \cdots \oplus 0 \oplus \mathbf{I}_j \oplus 0 \oplus \cdots \oplus 0,$$

where the $d_j \times d_j$ identity matrix $\mathbf{I}_j$ is in the jth position. The inverse Fourier transform of this vector is the function (evaluated at $g \in G$)

$$\frac{d_j}{|G|} \operatorname{tr}(\rho_j(g^{-1})) = \frac{d_j}{|G|} \chi_j(g^{-1}) = \frac{d_j}{|G|} \overline{\chi_j(g)}.$$

Therefore, for all $\phi \in \mathbb{C}^n$, we have

$$P_j(\phi) = \frac{d_j}{|G|}\tilde{\chi}_j * \phi,$$

where $\tilde{\chi}_j(g) = \text{tr}(\rho_j(g^{-1}))$ is the (inverted) character of the irreducible representation ρ_j. Let $\rho_j(k,l)(s)$ be the (k,l) entry in the $d_j \times d_j$ matrix $\rho_j(s)$ and consider the function defined by $f(s) = \rho_j(k,l)(s^{-1})$, $s \in G$. Observe the image and the kernel of P_j are given by

$$\text{Im}(P_j) = \text{span}\{\rho_j(k,l) \mid k,l \in 1, \ldots, d_j\}$$

and

$$\text{Ker}(P_j) = \text{Im}(P_j)^{\perp} = \text{span}\{\rho_i(k,l) \mid i \neq j \text{ and } k,l \in 1, \ldots, d_i\}.$$

Note $\dim\left(\text{Im}(P_j)\right) = d_j^2$. Moreover, the functions

$$\left\{\sqrt{\frac{d_j}{|G|}}\rho_j(k,l) \mid k,l \in 1, \ldots, d_j\right\}$$

form an orthonormal basis for $\text{Im}(P_j)$. Also note $\text{Im}(P_j) \perp \text{Im}(P_i)$ for $i \neq j$. We also have

$$\tilde{\chi}_j * \phi = \sum_{k,l}\langle\phi, \rho_j(k,l)\rangle\rho_j(k,l).$$

The important observation here is the fact that the space $\text{Im}(P_j)$ is invariant under the group circulant matrix $C = C_G(\psi)$ and is also ψ independent. We have an orthonormal basis for each $\text{Im}(P_j)$, and thus for $\mathbb{C}^n$, but these vectors no longer need to be eigenvectors for the group circulant matrix $C = C_G(\psi)$. Still, this orthonormal basis is ψ independent. For more information we refer the reader to Zizler (2013). For more details on group representations, we refer the reader to Dummit (1999). We refer the reader to Stankovic et al. (2005) or An and Tolimieri (2003) for more material on the non-abelian Fourier transform and its applications.

2. Main results

We need the orthogonal components $\{X_j\}$ to be real valued for real-life applications. An element of a group G is said to be real if it is conjugate to its inverse. Recall that two elements a and b of a group G are said to be conjugate if there exists $c \in G$ such that $c^{-1}ac = b$. A conjugacy class is said to be real if it has a real element. Note that if a conjugacy class has a real element then all the elements of this conjugacy class are real. It is a known result, see James and Liebeck (1993), for example, that the number of real irreducible characters of a group G is equal to the number of real conjugacy classes of G. Therefore, a group G has all irreducible characters real if and only if all the elements of that group are real. Therefore, the symmetric group S_n has all the irreducible characters real as all of its elements are real. However, this is no longer true for the alternating groups.

Recall r denotes the number of conjugacy classes for G and $\{\tilde{\chi}_j\}_{j=1}^r$ denotes the set of all the (inverted) irreducible characters of G. Let d_j denote the degree of the irreducible representation ρ_j. Define a vector valued zero mean WSS process over G as

$$X = (X(g_0), X(g_1), \ldots, X(g_{n-1}))^T,$$

where $G = \{g_0, g_1, \ldots, g_{n-1}\}$ and each $X(g_i)$ is a zero mean WSS process. Consider a decomposition of $X = \sum_{j=1}^{p} X_j$ where $\{X_j\}$ is a mutually orthogonal set of zero mean real-valued WSS processes. We say that the value p is optimal if it is impossible to decompose X into more zero mean real-valued mutually orthogonal WSS processes. Here, of course, the value p should be independent of X. Thus this decomposition is requested for all X and is only group G dependent. Note that a specific process X could potentially be decomposed into more mutually orthogonal components.

THEOREM 1 *Let X denote a zero mean real-valued WSS process over the group G. Let $p = \frac{r+s}{2}$ where s is the number of real conjugacy classes of G. Then, we can write*

$$X = \sum_{j=1}^{p} X_j,$$

where $\{X_j\}_{j=1}^{r}$ is a mutually orthogonal set of zero mean real-valued WSS processes and the value p is optimal. Here, we have

$$X_j = \frac{d_j}{|G|}\tilde{\chi}_j * X = \frac{d_j}{|G|}C_G(\tilde{\chi}_j)X \text{ if} \chi_j \text{ is a real}$$

and

$$X_j = \frac{d_j}{|G|}\left(\chi_j + \tilde{\chi}_j\right) * X = \frac{d_j}{|G|}C_G(\chi_j + \tilde{\chi}_j)X \text{ if } \chi_j \text{ is a complex.}$$

Moreover, we have

$$E(X(t)X^*(s)) = \sum_{j=1}^{p} E(X_j(t)X_j^*(s))$$

for all $t \in G$.

Proof Define the Fourier transforms of the two zero mean WSS processes X and Y evaluated at the irreducible representation ρ_j

$$\hat{X}\left(\rho_j\right) = \sum_{t\in G} X(t)\rho_j(t); \hat{Y}(\left(\rho_j\right)) = \sum_{t\in G} Y(t)\rho_j(t).$$

Consider the cross-correlation function R_{XY} and let $\mathcal{F}$ denote the Fourier transform operator. Define

$$C_j = \mathcal{F}(R_{XY})(\rho_j).$$

We have

$$\begin{aligned} E(\hat{X}\left(\rho_j\right)Y^*(s)) &= \sum_t E(X(t)Y^*(s))\rho_j(t) \\ &= \sum_t R_{XY}(ts^{-1})\rho_j(t) \\ &= \sum_t R_{XY}(\tau)\rho_j(\tau s) \\ &= C_j\rho_j(s) \end{aligned}$$

we obtain

$$\begin{aligned} E(\hat{X}\left(\rho_j\right)\hat{Y}^*\left(\rho_j\right)) &= \sum_s E(\hat{X}\left(\rho_j\right)Y^*(s))\rho_j(s^{-1}) \\ &= \sum_s C_j\rho_j(s)\rho_j(s^{-1}) \\ &= \sum_s C_j \\ &= |G|C_j. \end{aligned}$$

Thus

$$C_j = \frac{1}{|G|}E(\hat{X}\left(\rho_j\right)\hat{Y}^*\left(\rho_j\right)).$$

Define

$$X_j = \frac{d_j}{|G|}\tilde{\chi}_j * X = \frac{d_j}{|G|}C_G(\tilde{\chi}_j)X;\ Y_j = \frac{d_j}{|G|}\tilde{\chi}_j * Y = \frac{d_j}{|G|}C_G(\tilde{\chi}_j)Y.$$

Note, here the components X_j can be complex valued. Set $Z = X - \sum_{i=1}^{r} X_i$ and let R_Z denote the autocorrelation function for Z. Observe for any $j \in \{1, 2, \ldots, r\}$

$$E\left(\hat{X}_{i_1}\left(\rho_j\right)\hat{X}_{i_2}\left(\rho_j\right)\right) = 0 \text{ if } i_1 \neq i_2,$$

and consider, for any $j \in \{1, 2, \ldots, r\}$

$$\begin{aligned}\mathcal{F}(R_Z)(\rho_j) &= \frac{1}{|G|}\left(E\left(\hat{Z}\left(\rho_j\right)\hat{Z}^*\left(\rho_j\right)\right)\right)\\ &= \frac{1}{|G|}\left(E\left(\hat{X}\left(\rho_j\right)\hat{X}^*\left(\rho_j\right)\right) - 2\sum_{i=1}^{r}E\left(\hat{X}\left(\rho_j\right)\hat{X}_i^*\left(\rho_j\right)\right) + \sum_{i_1,i_2=1}^{r}E\left(\hat{X}_{i_1}\left(\rho_j\right)\hat{X}_{i_2}^*\left(\rho_j\right)\right)\right)\\ &= \frac{1}{|G|}\left(E\left(\hat{X}\left(\rho_j\right)\hat{X}^*\left(\rho_j\right)\right) - 2E\left(\hat{X}\left(\rho_j\right)\hat{X}^*\left(\rho_j\right)\right) + E\left(\hat{X}\left(\rho_j\right)\hat{X}^*\left(\rho_j\right)\right)\right)\\ &= 0.\end{aligned}$$

Now we have

$$\mathcal{F}\left(R_{X_iX_j}\right)(\rho_n) = 0$$

for all $n = \{1, 2, \ldots, r\}$ as long as $i \neq j$. Thus, we have, for $i \neq j$, $R_{X_iX_j}(\tau) = 0$ for all $\tau \in G$. In this manner, we have a decomposition into potentially complex components. We have $X = \sum_{j=1}^{r} X_j$ as zero mean WSS processes over G.

To analyze the real case, we make a key observation. If χ is an irreducible character of G then so is it conjugate. Note that $\chi(g) = \tilde{\chi}(g) = \chi(g^{-1}) = \overline{\chi(g)}$. Consider the complex mutually orthogonal decom*position as above*

$$X = \sum_{j=1}^{r} X_j.$$

If χ_j is a real-valued irreducible character we set $Y_j = X_j$ in the sum. If χ_{j_1} is complex valued, we pair it up with the irreducible representation $\overline{\chi_j}$ which is another irreducible character in the list of irreducible characters of G, say χ_{j_2}. In this case we set

$$Y_j = X_{j_1} + X_{j_2}.$$

Therefore, the value p equals to s, the total number of real irreducible characters of G, which equals to the number of real conjugacy classes, plus half of the remaining complex-valued irreducible characters of G. Thus $p = \frac{r+s}{2}$.

We now address the optimality of the decomposition. Observe that we are requesting, for all X and for all $n = \{1, 2, \ldots, r\}$

$$E\left(\hat{X}_{j_1}(\rho_n)\hat{X}_{j_2}(\rho_n)\right) = 0 \quad \text{for all } j_1 \neq j_2.$$

We can associate $\hat{X}_j$ with the following element in $M(\hat{G})$

$$\hat{X}_j \equiv 0 \oplus \ldots \oplus 0 \oplus \hat{X}_j \oplus 0 \oplus \ldots \oplus 0.$$

Suppose we can decompose an arbitrary X further. This would mean there exist square matrices E and F so that

$E + F = I$ and $EQFQ = 0$, for all square Q.

Assume Q is non-singular and we have $EQ(I - E) = 0$ which implies $EQ = EQE$. We re-write E as follows:

$E = EQEQ^{-1}$ for all Q.

Now choose Q so that QEQ^{-1} is J, the Jordan canonical form of E. Now it is straightforward to see that the relation $E = EJ$ is impossible unless $E = 0$ or $E = I$. The result now follows. □

Note that if all elements of the group are real then $p = r$. The above theorem can be used to decomposition the variance of $X(t)$, in particular, for any $t \in G$, we have

$$E(X(t)X^*(t)) = \sum_{j=1}^{p} E(X_j(t)X_j^*(t)).$$

The interpretation of $var(X_1) = E(X_1(t)X_1(t))$ is straightforward since ρ_0 denotes the trivial group representation. This quantity refers to the amount of variance of $X(t)$ captured by the variance of the mean of X over G. The above decomposition $X = \sum_{j=1}^{r} X_j$ into complex components will be referred to as the optimal complex orthogonal decomposition, similarly, the above decomposition $X = \sum_{j=1}^{p} X_j$ into real components will be referred to as the optimal real orthogonal decomposition.

THEOREM 2 *We have $E(X(t)X(s)) = 0$ for all $t, s \in G$ such that $t \neq s$ if and only if for any $t \in G$, we have*

$$var(X_j(t)) = \frac{d_j^2}{|G|} varX(t),$$

where $\{X_j\}_{j=1}^{r}$ is the optimal complex orthogonal decomposition.

Proof Consider the decomposition $X = \sum_{j=1}^{r} X_j$ into possibly complex valued components and suppose $E(X(t)X(s)) = 0$ for all $t \neq s$. Then, we have

$$\begin{aligned}
E\left(X_j(t)X_j(t)\right) &= E\left(\frac{d_j}{|G|}\left(C_G(\tilde{\chi}_j)X\right)(t)\frac{d_j}{|G|}\left(C_G(\tilde{\chi}_j)X\right)(t)\right) \\
&= \frac{d_j^2}{|G|^2}E\left(C_G(\tilde{\chi}_j)C_G(\tilde{\chi}_j)X(t)X(t)\right) \\
&= \frac{d_j^2}{|G|^2}E\left(C_G(\tilde{\chi}_j)X(t)X(t)\right) \\
&= \frac{d_j^2}{|G|^2}\,tr\left(C_G(\tilde{\chi}_j)\right)varX(t) \\
&= \frac{d_j^2}{|G|^2}|G|\,varX(t) \\
&= \frac{d_j^2}{|G|}varX(t).
\end{aligned}$$

Conversely, suppose $var(X_j(t)) = \frac{d_j^2}{|G|}\,varX(t)$ for all $t \in G$. Define $n = |G|$ unknowns $\{x_\tau\}_{\tau \in G}$, where $x_\tau = E(X(t)X(s)) = R_X(\tau)$ with $\tau = t^{-1}s$. Now consider the following linear equations,

$$\{E\left(C_G(\tilde{\chi}_j)X(t)X(t)\right) = |G|\,varX(t)\}_{j=\{1,\ldots,r\},\, t\in G}$$

and note that the number of these equations is equal to rn. However, we have only as many linearly independent equations as there are linearly independent columns in the matrices

$$\{C_G(\tilde{\chi}_j)\}_{j=1}^{r}.$$

This number is equals to $\sum_{j=1}^{r} d_j^2 = n$, the number of unknowns. The result follows with the unique solution $E(X(t)X(s)) = 0$ for $t \neq s$. □

As an application of the above result we develop a hypothesis testing for correlation of the WSS processes $X(t)$ and $X(s)$ based on testing equality of variances of the following possibly complex-valued random variables

$$\left\{ \frac{1}{d_j} X_j \right\}_{j=1}^{r}.$$

Testing the above random variables might be better that testing correlations of $X(t)$ and $X(s)$ due to averaging process that could bring the random variables close to normality.

3. The group S_3

We will consider the symmetric group S_3 in our example. The group $G = S_3$ consists of elements

$g_0 = (1);\ g_1 = (12);\ g_2 = (13)$
$g_3 = (23);\ g_4 = (123);\ g_5 = (132).$

The group S_3 has three conjugacy classes

$$\{g_0\},\ \{g_1, g_2, g_3\},\ \{g_4, g_5\}.$$

We have three irreducible representations, two of which are one-dimensional, ρ_1 is the identity map, ρ_2 is the map that assigns the value of 1 if the permutation is even and the value of -1 if the permutation is odd. Finally, we have ρ_3, the two-dimensional irreducible representation of S_3, defined by the following assignment

$$g_0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};\ g_1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$g_2 \mapsto \begin{pmatrix} 0 & e^{2\pi i/3} \\ e^{-2\pi i/3} & 0 \end{pmatrix};\ g_3 \mapsto \begin{pmatrix} 0 & e^{-2\pi i/3} \\ e^{2\pi i/3} & 0 \end{pmatrix}$$
$$g_4 \mapsto \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix};\ g_5 \mapsto \begin{pmatrix} e^{-2\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{pmatrix}.$$

The irreducible characters of S_3 are given by

$$\chi_1 = (1, 1, 1, 1, 1, 1)^T$$
$$\chi_2 = (1, -1, -1, -1, 1, 1)^T$$
$$\phi_3 = (2, 0, 0, 0, -1, -1)^T,$$

where χ_1 and χ_2 are also multiplicative characters. Moreover, we have

$$\rho_3(1, 1) = (1, 0, 0, 0, e^{-2\pi i/3}, e^{2\pi i/3})^T$$
$$\rho_3(1, 2) = (0, 1, e^{-2\pi i/3}, e^{2\pi i/3}, 0, 0)^T$$
$$\rho_3(2, 1) = (0, 1, e^{2\pi i/3}, e^{-2\pi i/3}, 0, 0)^T$$
$$\rho_3(2, 2) = (1, 0, 0, 0, e^{2\pi i/3}, e^{-2\pi i/3})^T.$$

The G-convolution by a function $\psi = (c_0, c_1, c_2, c_3, c_4, c_5)^T \in l^2(G)$ can be induced by a G-circulant matrix $C_G(\psi)$ given by

$$C_G(\psi) = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_1 & c_0 & c_4 & c_5 & c_2 & c_3 \\ c_2 & c_5 & c_0 & c_4 & c_3 & c_1 \\ c_3 & c_4 & c_5 & c_0 & c_1 & c_2 \\ c_4 & c_3 & c_1 & c_2 & c_5 & c_0 \\ c_5 & c_2 & c_3 & c_1 & c_0 & c_4 \end{pmatrix}$$

and specifically, note that $\tilde{\chi}_j(g) = \overline{\chi}_j(g) = \chi_j(g)$. We have

$$C_G(\chi_1) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

$$C_G(\chi_2) = \begin{pmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{pmatrix}.$$

$$C_G(\chi_3) = \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 2 \\ -1 & 0 & 0 & 0 & 2 & -1 \end{pmatrix}.$$

Set

$$\begin{aligned} X &= [X(g_0), X(g_1), X(g_2), X(g_3), X(g_4), X(g_5)]^T \\ &= [X(0), X(1), X(2), X(3), X(4), X(5)]^T \end{aligned}$$

and we obtain

$$X_1 = \frac{1}{6}C(\chi_1)X = \frac{1}{6}\langle X, \chi_1\rangle\chi_1.$$
$$X_2 = \frac{1}{6}C(\chi_2)X = \frac{1}{6}\langle X, \chi_2\rangle\chi_2.$$
$$X_3 = \frac{1}{3}C(\chi_3)X.$$

Therefore, the set $\{X_1, X_2, X_3\}$ is an optimal set consisting of real zero mean mutually orthogonal WSS processes over S_3. As a result we have

$$E(X(t)X^*(t)) = \sum_{j=1}^{r} E(X_j(t)X_j^*(t)) \quad \text{for all } t \in G.$$

4. Carry-over effects in the cross-over designs

The above orthogonal decomposition of the WSS processes appear naturally in the cross-over designs in clinical trials, in particular, the William's 6×3 design with 3 treatments. During a cross-over trial every patient receives more than one treatment in a certain pre-specified sequence. Thus, each

subject then acts as his or her own control. Each treatment is administered for a pre-selected time period. In these experiments a washout period is established between the last administration of one treatment and the first administration of the next treatment. The intention is for the effect of the preceding treatment to wear off during the trial. However, there will be some carry-over effects in all the specified treatment sequences, clearly starting with the second treatment. For more information on cross-over designs in clinical trials see Senn (2002) or Chow and Liu (2013), for example.

Consider a certain sequence of treatments. By a carry-over effect within these treatments we understand the total sum of all effects arising from all the treatments within this sequence. In our example, we follow the William's 6×3 design with 3 treatments *A*, *B*, and *C*, in particular, the underlying group is the symmetric group of three elements S_3. In particular, we have six treatment sequences *ABC*, *ACB*, *BAC*, *BCA*, *CAB*, *CBA*. For example, suppose the order of treatment administration is *BCA*, with *B* first. We decide to collect the sum of all carry-over effects of the treatments in this sequence (starting with the second one), X_{BCA}. We observe the sequence *BCA* as a permutation of the sequence *ABC* by the permutation $g_4 = (123)$, an element of the group S_3. Thus, we can write $X_{BCA} = X(g_4) = X(4)$. Similarly, a permutation sequence *ACB* would result in $X_{ACB} = X(g_3) = X(3)$.

We treat X as a zero mean WSS process, if necessary we can subtract the mean to ensure zero mean WSS process. In particular, we assume the variance of the cross-over effect is the same for all treatment sequences. Moreover, we assume $E(X(g_i)X(g_j))$ only depends on the sequence $g_i^{-1}g_j$, in another words, depends only on the permutation that gets us from the sequence g_i to the sequence g_j. However, note that the random processes $X(g_i)$ and $X(g_j)$ are in general dependent and, in general, might have different distributions for $i \neq j$.

We can now decompose the variance of the zero mean WSS process X over S_3 as follows. For all $t \in G$ we have

$$var(X(t)) = var(X_1(t)) + var(X_2(t)) + var(X_3(t)).$$

In this context, the real zero mean WSS processes over S_3, $\{X(t)\}_{t \in S_3}$, are un-correlated if and only if

$$\left\{X_1, X_2, \frac{1}{2}X_3\right\}$$

have the same variance.

5. The group S_4

The symmetric group of four elements has five conjugacy classes represented by

$$(1); (12); (123); (1234); (12)(34)$$

with sizes 1, 6, 8, 6, and 3, respectively. We have two irreducible characters of degree 1, namely the principal one and the one whose value at a permutation is its sign. There is one-degree two irreducible character χ_3 and two-degree 3 irreducible representations χ_4 and χ_5. The value of χ_4 at a permutation is the number of fixed points in that permutation minus one. Finally, we have $\chi_5 = \chi_4\chi_2$. For reference, see Dummit (1999). Find below a character table for S_4.

conj. classes	(1)	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	−1	1	−1	1
χ_3	2	0	−1	0	2
χ_4	3	1	0	−1	−1
χ_5	3	−1	0	1	−1

Suppose we have a cross-over design experiment with 4 treatments where we have all permutations of treatments allowed. We study the variance of the carry-over effects from the sequence of

treatments. Once, again, we define the carry-over effect for the treatment sequence as the sum of all the carry-over effects in that specific sequence. We can now decompose the variance of the zero mean WSS process X over S_4 as follows. We define

$$X_1 = \frac{1}{24}C(\chi_1)X = \frac{1}{6}\langle X, \chi_1\rangle \chi_1.$$
$$X_2 = \frac{1}{24}C(\chi_2)X = \frac{1}{6}\langle X, \chi_2\rangle \chi_2.$$
$$X_3 = \frac{1}{12}C(\chi_3)X.$$
$$X_4 = \frac{1}{8}C(\chi_4)X.$$
$$X_5 = \frac{1}{8}C(\chi_5)X.$$

and obtain

$$var(X(t)) = var(X_1(t)) + var(X_2(t)) + var(X_3(t)) + var(X_4(t)) + var(X_5(t)).$$

In this context, the zero mean WSS processes $\{X(t)\}_{t\in S_4}$ are un-correlated if and only is

$$\left\{X_1, X_2, \frac{1}{2}X_3, \frac{1}{3}X_4, \frac{1}{3}X_5\right\}$$

have the same variance.

6. The group A_4

The case of the alternating group A_4 is a little different as $r \neq p$. Here, we would still have 4 treatments, but only even permutations of the treatments are allowed in the cross-over design. We have three irreducible representations of degree 1 and one of degree 3, see Dummit (1999) for reference. We have 4 conjugacy classes represented by

(1); (12)(34); (123); (132)

of sizes 1, 3, 4, and 4, respectively. The irreducible characters are not necessarily real, find below the character table.

conj. classes	(1)	(12)(34)	(123)	(132)
χ_1	1	1	1	1
χ_2	1	1	ψ	ψ^2
χ_3	1	1	ψ^2	ψ
χ_4	3	−1	0	0

where $\psi = e^{\frac{2\pi i}{3}}$. We can now decompose the variance of the zero mean WSS process X over A_4 as follows. We do not have 4 summands in the decomposition as we do not have 4 real irreducible characters. We start with complex decomposition

$$X = X_1 + X_2 + X_3 + X_4$$

with

$$var(X(t)) = var(X_1(t)) + var(X_2(t)) + var(X_3(t)) + var(X_4(t)).$$

where

$$X_1 = \frac{1}{12}C(\chi_1)X = \frac{1}{12}\langle X, \chi_1\rangle \chi_1$$
$$X_2 = \frac{1}{12}C(\chi_2)X = \frac{1}{12}\langle X, \chi_2\rangle \chi_2$$
$$X_3 = \frac{1}{12}C(\chi_3)X = \frac{1}{12}\langle X, \chi_3\rangle \chi_3$$
$$X_4 = \frac{1}{4}C(\chi_4)X.$$

We now create a real decomposition of X consisting of three zero mean real-valued WSS processes as follows:

$$var(X(t)) = var(Y_1(t)) + var(Y_2(t)) + var(Y_3(t)).$$

where, letting $\chi = \chi_2 + \chi_3$,

$$\begin{aligned} Y_2 &= \frac{1}{12}C(\chi_2 + \chi_3)X \\ &= \frac{1}{12}C(\chi)X \\ &= \frac{1}{12}\langle X, \chi\rangle \chi \end{aligned}$$

and $Y_1 = X_1, Y_3 = X_4$. Observe the values of χ on the respective conjugacy classses of A_4 are given as follows:

conj. classes	(1)	(12)(34)	(123)	(132)
χ	2	2	−1	−1

In this context, the zero mean WSS processes $\{X(t)\}_{t\in A_4}$ are un-correlated if and only if

$$\left\{X_1, X_2, X_3, \frac{1}{3}X_4\right\}$$

have the same variance.

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References

An, M., & Tolimieri, R. (2003). *Group filters and image processing*. Psypher Press.
Bartoszynski, R., & Niewiadomska-Bugaj, M. (2008). *Probability and statistical inference* (2nd ed.). New York, NY: Wiley.
Chow, S.-C. & Liu, J.-P. (2013). *Design and analysis of clinical trials: Concepts and methodologies*. Hoboken, NJ: Wiley.
Diaconis, P. (1998). *Group representations in probability and statistics*. Hayward, CA: Institute of Mathematical Statistics.
Dummit, D. S., & Foote, R. N. (1999). *Abstract algebra*. New York, NY: John Wiley and Sons.
Giannakis, G. B. (1999). *Cyclostationary signal analysis, digital signal processing handbook*. Boca Raton, FL: CRC Press LLC.
Hazewinkel, M. (Ed.). (2001). *Peter-Weyl theorem*. Encyclopedia of mathematics: Springer. ISBN 978-1-55608-010-4.
James, G., & Liebeck, M. (1993). *Representations and characters of groups*. New York, NY: Cambridge University Press.
Peccati, G., & Pycke, J.-R. (2005). Decompositions of stochastic processes based on irreducible group representations. (p. 27).
Senn, S. (2002). *Cross-over trials in clinical research* (2nd ed.). Queensland: Wiley.
Stankovic, Radomir S., Moraga, Claudio, & Astola, Jaakko T. (2005). *Fourier analysis on finite groups with applications in signal processing and system design*. Hoboken, NJ: IEEE Press, John Wiley and Sons.
Zizler, P. (2013). On spectral properties of group circulant matrices. *PanAmerican Mathematical Journal, 23*(1), 1–23.
Zizler, P. (2014). Some remarks on the non-abelian fourier transform in crossover designs in clinical trials. *Applied Mathematics, 5*, 917–927.

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